Introduction to Modal Logics

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A *modality* is an expression that is used to *qualify the truth* of a judgment. It has received a wide range of application in philosophy: a modality can be

- *Alethic.* A proposition P is *necessarily/possibily* true;
- *Epistemic.* An agent *knows* P is true;
- *Doxastic.* A person *believes* P is true;
- *Deontic.* P is *ought/forbidden* to be true;
- *Temporal.* P *is now/will be* true.

1 Syntax & Semantics

Let Prop be a given set of *propositional variables.*

Definition 1A *(Syntax).* A **formula** ranges over a *proposition*, *falsehood*, *negation*, *implication*, and *necessity* (i.e. box).

$$A, B ::= p \in \mathsf{Prop} \mid \bot \mid A \to B \mid \Box A$$

In addition, we define the following abbreviations:

$$\neg A \stackrel{\text{def}}{=} A \to \bot$$
$$\Diamond A \stackrel{\text{def}}{=} \neg \Box \neg A$$
$$A \vee B \stackrel{\text{def}}{=} \neg A \to B$$
$$A \wedge B \stackrel{\text{def}}{=} \neg (A \to \neg B)$$

These abbreviations are established in classical logic. In an intuinitionistic setting, we would include these connetives as part of the syntax.

Definition 1B *(Kripke semantics).* A **Kripke frame** $\langle W, R \rangle$ consists of

- a nonempty *world* $W \neq \emptyset$, and
- an *accessibility relation* $R \subseteq W \times W$.

A **valuation** $V : \mathsf{Prop} \to \mathcal{P}(W)$ assigns each propsition $p \in \mathsf{Prop}$ to the set of worlds $V(p) \subseteq W$ for which p is true. Together, $\mathcal{M} = \langle W, R, V \rangle$ forms a **model** of modal logic.

Frames and valuations serve different purposes. A *frame* gives a general structure of possible interpretations. There is a correspondence between the axioms a modal logic satisfy and properties of the accessibility function (see Section 3). A *valuation* gives *one* possible interpretation of the logic, by defining what propositional variables are true in which world.

If we think of $\langle W, R \rangle$ as a graph, then V can be seen as a vertex coloring[1].

1.1 Satisfiability and Validity

Definition 11A. $\mathcal{M}, w \vDash A$ reads "the model $\mathcal{M} = \langle W, R, V \rangle$ **satisfies** a formula A at world $w \in W$" and is defined as follows:

$$\begin{aligned}
&\mathcal{M}, w \vDash p && \text{iff } w \in V(p) \\
&\mathcal{M}, w \nvDash \bot && \\
&\mathcal{M}, w \vDash A \to B && \text{iff } \mathcal{M}, w \nvDash A \text{ or } \mathcal{M}, w \vDash B \\
&\mathcal{M}, w \vDash \Box A && \text{iff for all } v, wRv \text{ implies } \mathcal{M}, v \vDash A
\end{aligned}$$

We can derive that

$$\mathcal{M}, w \vDash \Diamond A \text{ iff there exists } v \text{ such that } wRv \text{ and } \mathcal{M}, v \vDash A$$

Definition 11B. $\vDash A$ reads "A is **valid**" and holds iff $\mathcal{M}, w \vDash A$ in every model $\mathcal{M} = \langle W, V, R \rangle$ and every world $w \in W$.

Example 11A. **System K** is the most basic modal logic. In $\boldsymbol{K}$, we have the following theorem (called Axiom K).

$$\boldsymbol{K} : \Box(A \to B) \to \Box A \to \Box B$$

To show this $\boldsymbol{K}$ is valid, assume $\Box(A \to B)$ and $\Box A$ holds in a model $\langle W, V, R \rangle$ and world $w \in W$. Let v be accessible by w, then it is the case that $A \to B$ and A hold in v as well. Therefore so is B.

2 Axioms

The most famous axioms are the following:

$$\boldsymbol{T} : \Box A \to A$$

[1] Let's allow a vertex to have zero or more colors.

Axioms

$$D : \Box A \to \Diamond A$$
$$B : A \to \Box \Diamond A$$
$$4 : \Box A \to \Box \Box A$$
$$5 : \Diamond A \to \Box \Diamond A$$

By switching between A and $\neg A$ in the axioms, we can derive the following semantically equivalent axioms:

$$T : A \to \Diamond A$$
$$D : \neg \Box \bot$$
$$B : \neg \Box \neg \Box A \to A$$
$$5 : \neg \Box \neg \Box A \to \Box A$$

Lemma 2A. By transitivity of implication, we have that

- T implies D;
- $T \land 5$ implies B.

Example 2A *(Epistemic logic)*. In epistemic logic, $\Box A$ is interpreted as "I know that A is true".[2] Since knowledge has to be true, we always assume Axiom T[3], saying what I know is true.

The other axioms also have epistemic interpretations.

- Axiom 4 is *positive introspection*: *I know that I know what I know*. Or, its contrapositive, *I don't know what I don't know that I know*.
- Axiom 5 is *negative introspection*: *I know that I don't know what I don't know*. Or, *I know what I don't know that I don't know*.

Notice that since T with 5 implies $B \equiv \neg \Box \neg \Box A \to A$, negative introspection is suggesting a counterintuitive idea that truth can be implied by the ignorance on one's own ignorance. Therefore, one can argue that $S5$ is not a suitable logic for epistemic reasoning.

Example 2B *(Doxastic logic)*. We can weaken T in epistemic logic and adopt B instead. This alludes to **doxastic logic**: the logic of believes. In doxastic logic, a belief may not be true (thus we don't assume T), but it should be consistent (i.e. cannot believe in contradictions).

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[2]Usually we would have *multiple modalities* where $\Box_a A$ means "agent a knows A".
[3]You can remember it as the "Truth" axiom.

3 Modal Axioms and Frame Properties

The most beautiful theorem in modal logic is the correspondence between modal axioms and frame properties. In particular, we have the following:

Theorem 3A. Given a frame $\langle W, R\rangle$, each axiom below holds for every valuation iff R satisfies certain corresponding properties.

Name	Modal axiom	Frame property
T	$\Box A \to A$	R is reflexive
4	$\Box A \to \Box\Box A$	R is transitive
B	$A \to \Box \Diamond A$	R is symmetric
D	$\Box A \to \Diamond A$	R is serial: $\forall w. \exists v. wRv$
5	$\Diamond A \to \Box \Diamond A$	R is Euclidean: $\forall w, v, u. wRv \land wRu \to wRu$

Proof. The proofs from frame properties to modal axioms are very straightforward and constructive. From modal axioms to frame property, we can prove by contrapositive and construct a valuation in which the implication is not satisfied.

- (T) If R is not reflexive at some $w \in W$, then let $V : \mathsf{Prop} \to \mathcal{P}(W)$ defined by $p \mapsto W \setminus \{w\}$.
- (4) Let $w, v, u \in W$ and assume $wRvRu$. Suppose $w\not R u$, then the valuation $V := p \mapsto \{x \in W : wRx\}$ renders $\Box p$ true at w, but not $\Box\Box p$ because vRu and $u \notin V(p)$.
- (B) Suppose wRv but $v \not R w$. Let $V := p \mapsto \{w\}$. Then p is true at w, but for any world v accessible from w, there exists no world accessible from v in which p is true.
- (D) Consider the valuation $V := p \mapsto W$. Then R being not serial at w means $\Box p$ is true at w but not for $\Diamond p$.
- (5) Suppose wRv, wRu, but $v \not R u$. Let $V := p \mapsto \{u\}$. Then $\Diamond A$ is true at w because wRu, but not $\Box \Diamond A$ because no world accessible from v is true.

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