MLTT Judgments and Structural Rules

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1 Judgments

Definition 1A *(Judgments)*.

- Type A is an **well-formed type** in context Γ. We express this as
$$\Gamma \vdash A \text{ type}.$$
- Type A is **judgmentally equal** to type B in context Γ. We express this as
$$\Gamma \vdash A \doteq B.$$
- Term a is an **element of** type B in context Γ. We express this as
$$\Gamma \vdash a : A$$
- Term a and b are **judgmentally equal** elements of A. We express this as
$$\Gamma \vdash a \doteq_A b$$

In the next sections, we may write $\Gamma \vdash \mathcal{J}$ for an arbitrary *judgmental thesis* $\mathcal{J}$.

2 Structural Rules

Every type theory has a set of **structural rules**. They describe the structure of a context and does not mention any specific form of a type. In general, a type theory will consist of these rules:

Definition 2A *(Presuppositions)*. Certain judgments can only be made under some presuppositions. For instance, a being an element of A implies that A must be a type. The following rules spell out all the presuppositions.

$$\frac{\Gamma, x : A \vdash B \text{ type}}{\Gamma \vdash A \text{ type}} \qquad \frac{\Gamma \vdash A \doteq B}{\Gamma \vdash A \text{ type}} \qquad \frac{\Gamma \vdash A \doteq B}{\Gamma \vdash B \text{ type}}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash A \text{ type}} \qquad \frac{\Gamma \vdash a \doteq_A b}{\Gamma \vdash a : A} \qquad \frac{\Gamma \vdash a \doteq_A b}{\Gamma \vdash b : A}$$

Definition 2B *(Judgmental equality is an equivalence relation)*. We postulate that judgmental equality for types and terms are reflexive, trasitive, and symmetric. We omit the six rules for them.

Definition 2C *(Variable conversion)*. We can swap judgmentally equal types in an judgment.

$$\frac{\Gamma \vdash A \doteq A' \quad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, x : A', \Delta \vdash \mathcal{J}}$$

Definition 2D *(Substitution)*. We can substitute a variable by an element of the same type.

$$\frac{\Gamma \vdash a : A \quad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]}\,\mathrm{S}$$

Additionally, we postulate that judgmental equality can be derived by substituting judgmentally equal terms/types.

$$\frac{\Gamma \vdash a \doteq_A a' \quad \Gamma, x : A, \Delta \vdash B \text{ type}}{\Gamma, \Delta[a/x] \vdash B[a/x] \doteq B[a'/x]}$$

$$\frac{\Gamma \vdash a \doteq a' \quad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[a/x] \vdash b[a/x] \doteq_{B[a/x]} b[a'/x]}$$

Definition 2E *(Weakening)*. Weakening a context by a type A preserves the judgmental thesis.

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, \Delta \vdash \mathcal{J}}{\Gamma, x : A, \Delta \vdash \mathcal{J}}\,\mathrm{W}$$

Some logics (e.g. linear, relevant) drops weakening in order to reason about resource usage.

Definition 2F *(Generic elements)*. We postulate that the variables declared in the context are indeed elements. This rule is reminiscent of the identity function.

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x : A \vdash x : A}\,\delta$$

Structural Rules

There are several other intuitive structural rules that one may postulate, we show that many of them are derivable from what we have.

Lemma 2A *(Rename variables).* Variables can always be renamed to a new one.

$$\frac{\Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, x' : A, \Delta[x'/x] \vdash \mathcal{J}[x'/x]}$$

Proof.

$$\frac{\dfrac{\Gamma \vdash A \text{ type}}{\Gamma, x' : A \vdash x' : A} \qquad \dfrac{\Gamma \vdash A \text{ type} \quad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, x' : A, x : A, \Delta \vdash \mathcal{J}}}{\Gamma, x' : A, \Delta[x'/x] \vdash \mathcal{J}[x'/x]}$$

□

Lemma 2B *(Exchange).* Given weakening and substitution, we can swap the position of two variables in the context.

$$\frac{\Gamma \vdash B \text{ type} \quad \Gamma, x : A, y : B, \Delta \vdash \mathcal{J}}{\Gamma, y : B, x : A, \Delta \vdash \mathcal{J}}$$

Proof. By substitution and weakening, we can obtain

$$\Gamma, y : B, x : A, y' : B, \Delta[y'/y] \vdash \mathcal{J}[y'/y].$$

Then, since $\Gamma, y : B, x : A \vdash y : B$, we can substitute y' back to y to get

$$\Gamma, y : B, x : A, \Delta \vdash \mathcal{J}.$$

□

Lemma 2C *(Element conversion).* From variable conversion, we can also derive **element conversion**.

$$\frac{\Gamma \vdash A \doteq A' \quad \Gamma \vdash a : A}{\Gamma \vdash a : A'}$$

with the following congruence

Structural Rules

$$\frac{\Gamma \vdash A \doteq A' \quad \Gamma \vdash a \doteq_A b}{\Gamma \vdash a \doteq_{A'} b}$$